

## ON JORDAN DERIVATIONS OF TRIANGULAR ALGEBRAS

XUEHAN CHENG AND WU JING

ABSTRACT. In this short note we prove that every Jordan derivation of triangular algebras is a derivation.

## 1. INTRODUCTION

Suppose that  $\mathcal{A}$  is an algebra over a commutative ring  $\mathcal{R}$ . An  $\mathcal{R}$ -linear map  $\delta$  from  $\mathcal{A}$  to an  $\mathcal{A}$ -module  $\mathcal{M}$  is said to be a *derivation* if  $\delta(ab) = \delta(a)b + a\delta(b)$  for any  $a, b \in \mathcal{A}$ . We call  $\delta$  a *Jordan derivation* if  $\delta(a^2) = \delta(a)a + a\delta(a)$  for each  $a \in \mathcal{A}$ . Obviously, every derivation is a derivation. But the inverse, in general, is not true (see [1]).

It is natural and very interesting to find some conditions under which each Jordan derivation is a derivation. The first result on this direction was due to Herstein in 1957. He proved that every Jordan derivation on a 2-torsion free prime ring is a derivation. In 1988, Brešar generalized Herstein's result to Jordan derivations of semiprime rings ([2]). Recently, Zhang and Yu ([6]) showed that every Jordan derivation of a triangular algebra is a derivation. More precisely, they proved the following result.

**Theorem 1.1.** ([6]) *Let  $\mathcal{A}, \mathcal{B}$  be unital algebras over a 2-torsion free commutative ring  $\mathcal{R}$ , and  $\mathcal{M}$  be a unital  $(\mathcal{A}, \mathcal{B})$ -bimodule that is faithful as a left  $\mathcal{A}$ -module and also as a right  $\mathcal{B}$ -module. Then every Jordan derivation from the triangular algebra  $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$  into itself is a derivation.*

Note that this result requires that both  $\mathcal{A}$  and  $\mathcal{B}$  are unital, and the proof is heavily dependent on

$$\mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The aim of this short note is to generalize Theorem 1.1 to more general case. We want to mention here that we do not require the existence of identities for both  $\mathcal{A}$  and  $\mathcal{B}$  and our approach is quite different from that in [6].

---

*Date:* June 12, 2007.

*1991 Mathematics Subject Classification.* 16W25.

*Key words and phrases.* Jordan derivations; derivations; triangular algebras.

This work is partially supported by NNSF of China (No. 10671086) and NSF of Ludong University (No. LY20062704).

We now introduce some definitions and notations. Recall that a *triangular algebra*  $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$  is an algebra of the form

$$\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations, where  $\mathcal{A}$  and  $\mathcal{B}$  are two algebras over a commutative ring  $\mathcal{R}$ , and  $\mathcal{M}$  is an  $(\mathcal{A}, \mathcal{B})$ -bimodule which is faithful as a left  $\mathcal{A}$ -module and also as a right  $\mathcal{B}$ -module (see [3]).

Throughout this paper, we set

$$\mathcal{T}_{11} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathcal{A} \right\},$$

$$\mathcal{T}_{12} = \left\{ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} : m \in \mathcal{M} \right\},$$

and

$$\mathcal{T}_{22} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in \mathcal{B} \right\}.$$

Then we may write  $\mathcal{T} = \mathcal{T}_{11} \oplus \mathcal{T}_{12} \oplus \mathcal{T}_{22}$ , and every element  $a \in \mathcal{T}$  can be written as  $a = a_{11} + a_{12} + a_{22}$ . Note that notation  $a_{ij}$  denotes an arbitrary element of  $\mathcal{T}_{ij}$ .

## 2. JORDAN DERIVATION OF TRIANGULAR ALGEBRAS

Throughout this section,  $\mathcal{A}$  and  $\mathcal{B}$  will be two algebras over a 2-torsion free commutative ring  $\mathcal{R}$  with the property:

(P) Suppose that  $a \in \mathcal{A}$  (resp.  $\mathcal{B}$ ). If  $xay + yax = 0$  holds for all  $x, y \in \mathcal{A}$  (resp.  $\mathcal{B}$ ), then  $a = 0$ .

Map  $\delta$  will be a Jordan derivation from triangular algebra  $\mathcal{T} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$  into itself, where  $\mathcal{M}$  is a faithful  $(\mathcal{A}, \mathcal{B})$ -bimodule.

We begin with the following useful lemma.

**Lemma 2.1.** *Let  $a$  be in  $\mathcal{A}$  (resp.  $\mathcal{B}$ ).*

- (i) *If  $xax = 0$  for any  $x \in \mathcal{A}$  (resp.  $\mathcal{B}$ ), then  $a = 0$ ;*
- (ii) *If  $ax + xa = 0$  for any  $x \in \mathcal{A}$  (resp.  $\mathcal{B}$ ), then  $a = 0$ .*

*Proof.* (i) It follows directly from Property (P).

(ii) For arbitrary  $x, y \in \mathcal{A}$ , multiplying  $ax + xa = 0$  from the left and the right by  $y$  respectively and adding them together, we obtain

$$yax + xay + yxa + xya = 0.$$

This leads to

$$yax + xay - ayx - xya = 0.$$

Furthermore, we have

$$yax + xay + yax + xay = 0.$$

By Property (P), we see that  $a = 0$ . □

The following result is given by Herstein.

**Lemma 2.2.** *Let  $\delta$  be a Jordan derivation on a 2-torsion free ring  $\mathcal{R}$  into itself. For any  $a, b, c \in \mathcal{R}$ , the following hold.*

- (i)  $\delta(ab + ba) = \delta(a)b + a\delta(b) + \delta(b)a + b\delta(a)$ ;
- (ii)  $\delta(aba) = \delta(a)ba + a\delta(b)a + ab\delta(a)$ ;
- (iii)  $\delta(abc + cba) = \delta(a)bc + a\delta(b)c + ab\delta(c) + \delta(c)ba + c\delta(b)a + cb\delta(a)$ .

**Lemma 2.3.** *For arbitrary  $a_{11} \in \mathcal{T}_{11}$  and  $b_{22} \in \mathcal{T}_{22}$ , we have*

- (i)  $\delta(a_{11})_{22} = 0$ ;
- (ii)  $\delta(b_{22})_{11} = 0$ ;
- (iii)  $\delta(a_{11}b_{22}) = \delta(a_{11})b_{22} + a_{11}\delta(b_{22})$ ;
- (iv)  $\delta(b_{22}a_{11}) = \delta(b_{22})a_{11} + b_{22}\delta(a_{11})$ .

*Proof.* We compute

$$\begin{aligned}
 0 &= \delta(a_{11}b_{22} + b_{22}a_{11}) \\
 &= \delta(a_{11})b_{22} + a_{11}\delta(b_{22}) + \delta(b_{22})a_{11} + b_{22}\delta(a_{11}) \\
 &= \delta(a_{11})_{12}b_{22} + \delta(a_{11})_{22}b_{22} + a_{11}\delta(b_{22})_{11} \\
 &\quad + a_{11}\delta(b_{22})_{12} + \delta(b_{22})_{11}a_{11} + b_{22}\delta(a_{11})_{22}.
 \end{aligned}$$

It follows that

$$\delta(a_{11})_{22}b_{22} + b_{22}\delta(a_{11})_{22} = 0$$

and

$$a_{11}\delta(b_{22})_{11} + \delta(b_{22})_{11}a_{11} = 0.$$

By Lemma 2.1, we have  $\delta(a_{11})_{22} = \delta(b_{22})_{11} = 0$ .

Now (iii) and (iv) follows easily. □

**Lemma 2.4.** *For any  $a_{12} \in \mathcal{T}_{12}$ , the following are true.*

- (i)  $\delta(a_{12})_{11} = 0$ ;
- (ii)  $\delta(a_{12})_{22} = 0$ .

*Proof.* For arbitrary  $b_{11} \in \mathcal{T}_{11}$ , we have

$$\begin{aligned}
 0 &= \delta(b_{11}a_{12}b_{11}) \\
 &= \delta(b_{11})a_{12}b_{11} + b_{11}\delta(a_{12})b_{11} + b_{11}a_{12}\delta(b_{11}) \\
 &= b_{11}\delta(a_{12})_{11}b_{11} + b_{11}a_{12}\delta(b_{11})_{22} \\
 &= b_{11}\delta(a_{12})_{11}b_{11}.
 \end{aligned}$$

It follows from Lemma 2.1 that  $\delta(a_{12})_{11} = 0$ .

Similarly, we can get  $\delta(a_{12})_{22} = 0$  by considering  $\delta(b_{22}a_{12}b_{22})$  for any  $b_{22} \in \mathcal{T}_{22}$ . □

**Lemma 2.5.** *For any  $a_{11} \in \mathcal{T}_{11}$  and  $b_{12} \in \mathcal{T}_{12}$ ,*

- (i)  $\delta(b_{12}a_{11}) = \delta(b_{12})a_{11} + b_{12}\delta(a_{11});$
- (ii)  $\delta(a_{11}b_{12}) = \delta(a_{11})b_{12} + a_{11}\delta(b_{12}).$

*Proof.* (i) Note that

$$\begin{aligned}\delta(b_{12})a_{11} + b_{12}\delta(a_{11}) &= \delta(b_{12})_{11}a_{11} + b_{12}\delta(a_{11})_{22} \\ &= 0 = \delta(b_{12}a_{11}).\end{aligned}$$

(ii) We have

$$\begin{aligned}\delta(a_{11}b_{12}) &= \delta(a_{11}b_{12} + b_{12}a_{11}) \\ &= \delta(a_{11})b_{12} + a_{11}\delta(b_{12}) + \delta(b_{12})a_{11} + b_{12}\delta(a_{11}) \\ &= \delta(a_{11})b_{12} + a_{11}\delta(b_{12}).\end{aligned}$$

□

Similarly, we have

**Lemma 2.6.** *For arbitrary  $a_{12} \in \mathcal{T}_{12}$  and  $b_{22} \in \mathcal{T}_{22}$ ,*

- (i)  $\delta(a_{12}b_{22}) = \delta(a_{12})b_{22} + a_{12}\delta(b_{22});$
- (ii)  $\delta(b_{22}a_{12}) = \delta(b_{22})a_{12} + b_{12}\delta(a_{12}).$

**Lemma 2.7.**  *$\delta$  is a derivation on  $\mathcal{T}_{12}$ .*

*Proof.* For any  $a_{12}, b_{12} \in \mathcal{T}_{12}$ , by Lemma 2.4, we have

$$\begin{aligned}\delta(a_{12})b_{12} + a_{12}\delta(b_{12}) &= \delta(a_{12})_{11}b_{12} + a_{12}\delta(b_{12})_{22} \\ &= 0 = \delta(a_{12}b_{12}).\end{aligned}$$

□

**Lemma 2.8.**  *$\delta$  is a derivation on  $\mathcal{T}_{11}$ .*

*Proof.* Let  $a_{11}, b_{11} \in \mathcal{T}_{11}$ , and  $c_{12} \in \mathcal{T}_{12}$  be arbitrary. On one side, we have

$$\begin{aligned}\delta(a_{11}b_{11}c_{12}) &= \delta(a_{11})b_{11}c_{12} + a_{11}\delta(b_{11}c_{12}) \\ &= \delta(a_{11})b_{11}c_{12} + a_{11}\delta(b_{11})c_{12} + a_{11}b_{11}\delta(c_{12}).\end{aligned}$$

On the other side, we get

$$\delta(a_{11}b_{11}c_{12}) = \delta(a_{11}b_{11})c_{12} + a_{11}b_{11}\delta(c_{12}).$$

Then we can infer that

$$\delta(a_{11}b_{11})c_{12} = \delta(a_{11})b_{11}c_{12} + a_{11}\delta(b_{11})c_{12}.$$

This yields that  $\delta(a_{11}b_{11}) = \delta(a_{11})b_{11} + a_{11}\delta(b_{11})$  since  $\mathcal{M}$  is a faithful left  $\mathcal{A}$ -module.

□

In the similar manner, one can get

**Lemma 2.9.**  $\delta$  is a derivation of  $\mathcal{T}_{22}$ .

Now we can get our main result of this note.

**Theorem 2.10.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two algebras over a 2-torsion free commutative ring  $\mathcal{R}$  with the property:

(P) Suppose that  $a \in \mathcal{A}$  (resp.  $\mathcal{B}$ ). If  $xay + yax = 0$  holds for all  $x, y \in \mathcal{A}$  (resp.  $\mathcal{B}$ ), then  $a = 0$ .

Let  $\mathcal{M}$  be a faithful  $(\mathcal{A}, \mathcal{B})$ -bimodule and  $\mathcal{T}$  be the triangular algebra  $\text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ . Then every Jordan derivation  $\delta$  on  $\mathcal{T}$  into itself is a derivation.

*Proof.* It follows from Lemmas 2.3-2.9. □

We end this note with the following remark.

**Remark 2.11.** It is easy to see that Theorem 1.1 is a special case of Theorem 2.10 when both  $\mathcal{A}$  and  $\mathcal{B}$  are unital. In other words, our result is a generalization of Theorem 1.1.

## REFERENCES

1. D. Benkovič, Jordan derivations and antiderivations on triangular matrices, *Linear Algebra Appl.* **397** (2005), 235–244.
2. M. Brešar, Jordan derivations on semiprime rings, *Proc. Amer. Math. Soc.*, **104** (1988), 1003–1006.
3. W. S. Cheung, Commuting maps on triangular algebras, *J. London Math. Soc.*, **63** (2001), 117–127.
4. P. Ji, Jordan maps on triangular algebras, *Linear Algebra Appl.*, (to appear).
5. W. S. Martindale III, When are multiplicative mappings additive? *Proc. Amer. Math. Soc.*, **21** (1969) 695–698.
6. J. Zhang, W. Yu, Jordan derivations of triangular algebras, *Linear Algebra Appl.*, **419** (2006), 251–255.

COLLEGE OF MATHEMATICS AND INFORMATION, LUDONG UNIVERSITY, YANTAI, 264025 P. R. CHINA

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FAYETTEVILLE STATE UNIVERSITY, FAYETTEVILLE, NC 28301

*E-mail address:* wjing@uncfsu.edu